

Kinetic k-essence and Quintessence

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Dark energy models with non-canonical kinetic energy terms, k-essence, can have dynamical and sound speed properties distinct from canonical scalar fields, quintessence. Concentrating on purely kinetic term Lagrangians, which can be technically natural, we investigate limits on the equation of state dynamics and sound speed behaviors and the extent to which these models can be separated from quintessence.

I. INTRODUCTION

In the quest for the physical origin of the cosmic acceleration, we have relatively little guidance from basic principles. For dynamical scalar field models, e.g. quintessence, one must posit a potential, ideally possessing naturalness, without fine tuning. However the potential receives quantum loop corrections from high energy physics, raising the problem of preservation of form, or technical naturalness. An alternative approach is adopting non-canonical kinetic terms in the scalar field Lagrangian, leading to k-essence models.

A subset of these – kinetic k-essence – offers the possibility of removing the potential altogether (or keeping it constant). With a shift symmetry in the field, $\phi(x) \rightarrow \phi(x) + \theta$, this provides technical naturalness, an important virtue for a physical theory, and so kinetic k-essence models are worth investigating. Furthermore, these are in a sense as simple as quintessence in that they also involve a single function, here $\mathcal{L} = F(X)$ where X is the kinetic energy.

In this article we study the interplay between equations of state $w(a)$, defined in terms of an effective pressure to energy density ratio $w = p/\rho$ or equivalently cosmic expansion dynamics, and kinetic k-essence Lagrangians possessing certain stability properties. This will enable us to establish to what extent cosmological expansion or distance data could encounter degeneracies in the interpretation of the physical origin of the cosmic acceleration. For example, what are the characteristics of a quintessence model appearing degenerate with a k-essence model, in the sense that they produce the same equation of state over some redshift range. We derive limits on this degeneracy by showing which regions of the w - w' phase space (where $w' = dw/d\ln a$, with a the expansion scale factor) k-essence can lie in (see [1, 2, 3] for analyses of other DE models). We will also consider the converse issue by going from some specific equations of state to the corresponding k-essence Lagrangians.

In §II we discuss some of the motivations for considering k-essence as a possible physical model for the origin of the cosmic acceleration. §III presents a pedagogic overview and explanation of some of the main properties of (purely kinetic) k-essence. In §IV, we analyze the stability of solutions and from this derive the condition the DE equation of state must satisfy for it to be degenerate

with a stable kinetic k-essence model. In §V, we give the closed form solution for this degeneracy condition and present a simple prescription for predicting the behavior of any kinetic k-essence Lagrangian. In §VI we exhibit the results for several illustrative examples.

II. MOTIVATION

About a decade ago, analysis of the luminosity-redshift relation of distant supernovae [4, 5] suggested that the expansion of the universe is accelerating. This has led people to postulate the existence of a mysterious new component to the universe called dark energy (DE), characterized by a negative equation of state $w = p/\rho$. Since then, many observations of different kinds, most notably of the cosmic microwave background [6] and large scale structure [7], have consolidated the picture of a spatially flat universe with contributions to the critical energy density of about 1/4 by matter (dark matter plus baryonic matter) and 3/4 by dark energy. One of the biggest challenges in modern physics has become to explain the nature of this dark energy.

A standard explanation of DE is Einstein's cosmological constant Λ , a constant uniform background energy density corresponding to an equation of state $w = -1$. The physical origin of Λ could perhaps be a vacuum energy from particle physics. A natural value for this vacuum energy density would be either of order $\rho_{\text{vac}} \approx M_{\text{fund}}^4$, where M_{fund} is some fundamental mass/energy scale in nature like the Planck scale M_p or perhaps the electroweak scale M_{EW} , or exactly zero if some symmetry (like supersymmetry) enforces it. However, the dark energy density is about 3/4 of the current critical energy density of the universe, giving $\rho_\Lambda \approx 10^{-122} M_p^4 \approx 10^{-54} M_{EW}^4$, which is outrageously small. This problem of the dark energy being so small (but not zero) compared to the natural physical scales is called the Cosmological Constant Problem.

Another fine-tuning problem related to the cosmological constant model is the fact that the dark energy and matter components are comparable in size today. If the dark energy truly is a cosmological constant, there is only a short period of time in the evolution of the universe when this is the case so why is it happening exactly now, while we are around to observe it? This issue is part of

the so called Coincidence Problem.

Even though a cosmological constant is in good agreement with the current data, the problems above provide ample motivation to look for DE models beyond a cosmological constant. The most popular alternatives to Λ are scalar field models, in particular quintessence (see [3, 8] and references therein), which describes a single scalar field ϕ with a standard Lagrangian density $\mathcal{L} = X - V(\phi)$, with $X = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. Quintessence models can describe a range of equations of state necessary for an accelerated expansion, but like Λ suffer from fine-tuning issues.

More recently, scalar field models with non-canonical kinetic energy have gained interest. These so called k-essence (the “k” standing for kinetic) models are described by Lagrangians of the general form [9] $\mathcal{L} = v(\phi)F(X) - V(\phi)$ (a canonical kinetic energy is given by $F(X) = X$ and $v(\phi) = 1$). K-essence was originally proposed as a model for inflation [10], and then as a model for dark energy [11], along with explorations of unifying dark energy and dark matter [12, 13]. It now appears increasingly likely from both theoretical stability issues and observational constraints (e.g. [14, 15, 16]) from matter clustering properties (dark matter is very clumpy while DE is quite smooth out to the Hubble scale) that dark matter and dark energy are not the same substance; we will treat k-essence purely as a dark energy candidate. One reason for the interest in k-essence is that it admits solutions that track the equation of state of the dominant type of matter (in the early universe this is radiation) until pressure-less matter becomes dominant, at which point the k-essence begins to evolve toward cosmological constant behavior [11, 17, 18]. Such behavior can to a certain degree solve the fine-tuning problems mentioned above.

A good way to look at k-essence is as a generalization of canonical scalar field models (i.e. quintessence). Let us consider the quintessence Lagrangian in a bit more detail. Where the fact that it must be a function only of X and ϕ comes from Lorentz invariance, the reason for the kinetic energy term being equal to X (i.e. canonical) is merely that we assume X to be small compared to some energy scale and higher order terms to be irrelevant. Even though this is often a correct assumption because of the Hubble damping, there exist cases where it is not. In k-essence, we look at models where the higher order terms are not necessarily negligible. This can give rise to interesting new dynamics not possible in quintessence.

Another motivation for studying k-essence is the relation of scalar field theory to the quantum mechanics of a single particle (see also [19]). Heuristically, field theory can be viewed as the continuum limit of a grid of particles, with the field at a certain point describing the excitation of the particle at that point. The canonical scalar field theory Lagrangian density can in that picture be seen as the generalization of the Lagrangian of a non-relativistic point particle $L = \frac{1}{2}m\dot{q}^2$ (where q is the particle’s position). The Lagrangian of a relativistic point particle $L = -m\sqrt{1 - \dot{q}^2}$ on the other hand

leads to a non-canonical field theory Lagrangian density $\mathcal{L} = -\sqrt{1 - 2X}$, i.e. a k-essence Lagrangian.

A third motivation for the study of k-essence is that non-canonical Lagrangians appear naturally in string theory. In particular, the tachyon effective Lagrangian ([20], also see [21], in particular section 8, for a very readable review) has the Dirac-Born-Infeld-like form $\mathcal{L} = -V(\phi)\sqrt{1 - X}$. Here, the field ϕ represents the tachyon condensate describing the evaporation of a D-brane. The Nambu-Goto action for a p -brane embedded in a $p + 2$ dimensional space-time can be written in the same form [22], except that ϕ has the role of the coordinate transverse to the brane in this scenario. We discuss this in a little more detail in the Appendix.

Finally, as mentioned in the Introduction, technical naturalness as from a shift symmetry gives an advantage for purely kinetic k-essence Lagrangians, and these involve only a single function, $\mathcal{L} = F(X)$, like quintessence.

III. THE MODEL

We study k-essence, dark energy described by a single, real scalar field ϕ , minimally coupled but with a non-canonical kinetic term. In general, the k-essence action is of the form

$$S = \int d^4x \sqrt{-g} F(\phi, X), \quad (1)$$

where $X := \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. We concentrate on the subclass of kinetic k-essence, with a ϕ -independent action

$$S = \int d^4x \sqrt{-g} F(X). \quad (2)$$

We assume a Friedmann-Robertson-Walker metric $ds^2 = dt^2 - a^2(t) d\vec{x}^2$ (where $a(t)$ is the scale factor) and work in units $c = \hbar = 1$. Unless explicitly stated otherwise, we assume ϕ to be smooth on scales of interest so that $X = \frac{1}{2}\dot{\phi}^2$. Note that this implies $X \geq 0$.

Varying the action (2) with respect to the metric gives the energy momentum tensor of the k-essence

$$T^{\mu\nu} = F_X \partial^\mu\phi \partial^\nu\phi - g^{\mu\nu} F, \quad (3)$$

where a subscripted X denotes differentiation with respect to X . Using that for a comoving perfect fluid the energy momentum tensor is given by $T_{\mu\nu} = -pg_{\mu\nu} + (\rho + p)\delta_\mu^0\delta_\nu^0$, the k-essence energy density ρ and pressure p are

$$\rho = 2XF_X - F \quad (4)$$

and

$$p = F. \quad (5)$$

Throughout this paper, we will assume that the energy density is positive so that $2XF_X - F > 0$. The equation of state is

$$w = \frac{p}{\rho} = \frac{F}{2XF_X - F}. \quad (6)$$

The equation of motion for the field can be found either by applying the Euler-Lagrange equation for the field to the action (2), or by plugging the energy density and pressure given above into the continuity equation for a perfect fluid. Either way, the result is

$$F_X \ddot{\phi} + F_{XX} \dot{\phi}^2 \ddot{\phi} + 3H F_X \dot{\phi} = 0, \quad (7)$$

or equivalently, in terms of X ,

$$(F_X + 2F_{XX}X)\dot{X} + 6H F_X X = 0, \quad (8)$$

where a dot denotes differentiation with respect to t and $H = \dot{a}/a$ is the Hubble parameter. This equation can be integrated to give

$$X F_X^2 = k a^{-6}, \quad (9)$$

with $k \geq 0$ a constant [23].

Note that equation (9) tells us that the possible solutions $X(a)$, and therefore the behavior of all physical properties of the k-essence (like ρ , p and w) *as a function of the scale factor*, are completely determined by the function $F(X)$ and do not depend on the evolution of the other types of energy density. The only dependence of the k-essence component on other components enters through $a(t)$. One consequence of this is to preclude the possibility of tracking solutions [24] that automatically follow the equation of state of the dominant form of matter in the universe. Tracking behavior is possible in general k-essence models that do have ϕ -dependence in the action [11, 17, 18].

An interesting distinction when discussing dark energy models is between dark energies with $w > -1$ and those with $w < -1$. The latter are referred to as phantom dark energy [25] and can have rather exotic properties. For instance, their energy density is an increasing function of the scale factor, which can be seen from the Friedmann equation $d \ln \rho / d \ln a = -3(1+w)$ (see [26] for a discussion of problems arising in phantom k-essence theories). The boundary between the phantom and non-phantom regime is $w = -1$, e.g. a time independent cosmological constant. If a DE evolves from one regime to another, this is called phantom crossing, but [27], e.g., showed that this is impossible for a purely kinetic k-essence. We refer to [28, 29] for a discussion of phantom crossing in the context of other DE models.

For kinetic k-essence, one can use equation (6) to express the condition $w > -1$ ($w < -1$) as a condition on the function $F(X)$. We need to consider the two possibilities $F > 0$ and $F < 0$ separately. In the first case, demanding the energy density be positive immediately implies $w > 0$. For $F < 0$, a positive energy density means that $2X F_X / F < 1$ so

$$w = \frac{-1}{1 - 2X F_X / F} > -1 \quad (10)$$

when $F_X > 0$. All together, the conditions become (cf. [26])

$$F > 0 \implies w > 0, \quad (11)$$

$$F < 0 \quad \& \quad F_X > 0 \implies w > -1, \quad (12)$$

$$F < 0 \quad \& \quad F_X < 0 \implies w < -1. \quad (13)$$

(Recall the condition $\rho > 0$, or $F_X > F/(2X)$, is implicit).

We conclude this section by pointing out a useful scaling property of the kinetic k-essence Lagrangian, namely $F(X) \rightarrow CF(BX)$ with $B > 0$ and C arbitrary constants, leaves the physical properties, such as equation of state $w(a)$, unchanged. In other words, once we have found an $F(X)$ that reproduces an equation of state of interest, we are free to rescale both X and F without affecting $w(a)$. The freedom to rescale F by a factor C follows from the fact that both p and ρ are proportional to F ; C could play the role of a constant potential (see §VID). The freedom to rescale X comes from the fact that one can always redefine the field $\phi \rightarrow \phi/\sqrt{B}$ without changing the physics. The reason we mention this property is because in the following we will often use it to rescale $F(X)$ into a convenient form (for example with $X = 1$ and $F = -1$ at redshift zero) or to leave out multiplicative constants in expressions for X or F .

IV. RESTRICTIONS ON $w(a)$ FROM STABILITY

In this section we restrict possible equations of state by demanding that the k-essence be stable against spatial perturbations. Since the k-essence action only depends on X and not on ϕ , the relevant quantity for determining whether or not a solution of the equation of motion (9) is stable is the adiabatic sound speed squared

$$c_s^2 := \frac{p_X}{\rho_X} = \frac{F_X}{2X F_{XX} + F_X} = \frac{F_X^2}{(X F_X^2)_X}. \quad (14)$$

Perturbations can become unstable if the sound speed is imaginary, $c_s^2 < 0$, so we insist on $c_s^2 > 0$, or equivalently [26]

$$(X F_X^2)_X > 0. \quad (15)$$

Writing the condition $c_s^2 > 0$ in terms of $w(a)$, using $c_s^2 = (dp/da)/(d\rho/da)$ (which is valid as long as $dX/da \neq 0$), places a restriction on the equations of state $w(a)$ that can be described by stable k-essence solutions. Using

$$\frac{d\rho}{da} = -\frac{3(1+w)}{a}\rho \quad (16)$$

and

$$\frac{dp}{da} = \frac{dw}{da}\rho + w \frac{d\rho}{da} = \frac{a(dw/da) - 3w(1+w)}{a}\rho, \quad (17)$$

we get

$$c_s^2 = \frac{dp/da}{d\rho/da} = \frac{3w(1+w) - w'}{3(1+w)} > 0, \quad (18)$$

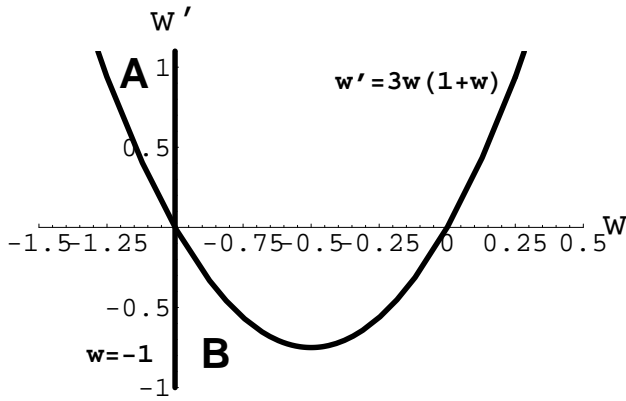


FIG. 1: The requirement that $c_s^2 > 0$ restricts the equation of state to the two regions A and B, bounded by the lines $w = -1$ and $w' = 3w(1+w)$.

where $w' := dw/d\ln a$. This restricts the equation of state to lie within two regions of the $w-w'$ plane (see Figure 1), bounded by the lines $w = -1$ and $w' = 3w(1+w)$. The first line separates phantom k-essence ($w < -1$) from ordinary k-essence and the second is the constant pressure line, as can be seen from Eq. (17). As mentioned previously, it is not possible to cross between regions A and B [27]. As we will show later, the requirement that $w(a)$ lies in region A or B implies that a number of popular ansatzes for $w(a)$ cannot realistically describe (stable) k-essence.

Note that a combination of stable kinetic k-essence models stays in the stable region [3] (and a combination of unstable models stays unstable). However, Λ plus a matter component (or more generally a $w \geq 0$ component; also see [30]) can look like stable k-essence. Another potentially interesting requirement to consider is $c_s^2 \leq 1$, which says that the sound speed should not exceed the speed of light, which suggests violation of causality. This condition would add an extra line to Figure 1, given by

$$w' = -3(1 - w^2). \quad (19)$$

However, we will not impose this condition because even though $c_s^2 > 1$ means signals can travel faster than light, this does not appear to lead to causal paradoxes (see for example [31]).

V. DISTINGUISHING KINETIC K-ESSENCE FROM QUINTESSENCE

If we know which equations of state can be reproduced by kinetic k-essence, we may be able to distinguish it observationally from other DE candidates. In the previous section, we derived a necessary condition on the DE equation of state for it to be described by a stable k-essence solution. In this section, we will answer a different question, namely what equations of state can be described by

k-essence in the first place? We will find that condition (18) from the previous section is in fact a sufficient condition. Any equation of state that stays within region A or B of the $w-w'$ plane can in principle be described by a k-essence Lagrangian density $F(X)$. Moreover, we will show how to construct this function. Our method consists of finding F and X as a function of the scale factor by expressing them in terms of physical quantities that can in principle be measured. These functions can then be used to construct $F(X)$. The first part is well defined for any dark energy, i.e. we can always formally construct $X(a)$ and $F(a)$ once we know, say, $w(a)$. However, if the slope of $X(a)$ changes sign, one ends up with a double valued function $F(X)$. In those cases, the dark energy cannot be described by a well defined k-essence model. We will see that double valuedness is produced exactly when crossing one of the lines $w = -1$ or $w' = 3w(1+w)$ in the $w-w'$ plane.

Now let us find $F(a)$ and $X(a)$. Equation (5) tells us that F is simply equal to the pressure

$$F(a) = p(a). \quad (20)$$

Inserting this into equation (6) and solving for XF_X gives

$$XF_X = \frac{1+w}{2w}p, \quad (21)$$

which squares to

$$X(XF_X^2) = \left(\frac{1+w}{2w}\right)^2 p^2. \quad (22)$$

Upon insertion of equation (9), this leads to

$$X(a) = Ca^6(\rho(a) + p(a))^2, \quad (23)$$

where $C > 0$ is a constant we can freely choose. Equations (20) and (23) define effective quantities for F and X even if there is no actual k-essence. If a well defined k-essence Lagrangian does exist, F and X have their standard meaning as the Lagrangian and as $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ respectively.

It is useful to find an expression for $X(a)$ in terms of the equation of state $w(a)$. Unfortunately, since ρ and p are in general integrals of $w(a)$, it is not possible to find a closed expression. However, it is possible to construct a differential equation. Differentiating equation (23) with respect to a and using equation (16) gives

$$\frac{dX}{d\ln a} = -6 \left(\frac{3w(1+w) - w'}{3(1+w)} \right) X = -6c_s^2 X. \quad (24)$$

For $F(a)$, we have the equation

$$\frac{dF}{d\ln a} = -\frac{3w(1+w) - w'}{w} F. \quad (25)$$

Given some evolution of the equation of state, we can derive $X(a)$ using Eq. (24) and then use Eq. (25) to find

$F(X)$. Alternatively, inverting the $X(a)$ from Eq. (24), one can use Eqs. (6) and (9) to get

$$F(X) \propto \frac{w(a(X))}{a^3(X)(1+w(a(X)))} \sqrt{X}. \quad (26)$$

From the dynamics in the w - w' plane, one can predict what sort of k-essence solution this corresponds to. If the dynamics crosses a boundary defining the four regions in Fig. 1, then c_s^2 changes sign, indicating a pathology within the k-essence picture. Note from Eq. (24) that c_s^2 changing sign corresponds precisely to the slope of $X(a)$ changing sign, and at the same time $F(X)$ becomes double valued. However, any equation of state that stays within one of the four regions can in principle be obtained from a k-essence Lagrangian. The stability argument from Section IV selects regions A and B from those four regions. In conclusion, imposing stability, k-essence can generate precisely the equations of state that lie in regions A and B of the w - w' plane.

Conversely, one can look at a given $F(X)$ and determine whether it is a viable k-essence model and what a corresponding quintessence model would be like. (Note that [32] considered a similar question in terms of the effective quintessence potential.) First, from the slope of the function one can deduce by applying Eqs. (11)-(13) whether the model is phantom or not. Since the k-essence field cannot cross $w = -1$ then the function $F(X)$ cannot change the sign of its slope, and it cannot be double valued (requiring the slope to go infinite). From the curvature of the function (concave or convex), in combination with the slope, one can read off whether the adiabatic sound speed is real or imaginary, and hence look for stability:

$$w > -1 : \quad F_{XX} > -\frac{F_X}{2X}, \quad [\text{suff. } F_{XX} \geq 0] \quad (27)$$

$$w < -1 : \quad F_{XX} < -\frac{F_X}{2X}, \quad [\text{suff. } F_{XX} \leq 0]. \quad (28)$$

Here suff. indicates a sufficient (but not necessary) condition for stability; this also corresponds to $0 < c_s^2 \leq 1$.

Since by eye one can usually tell when the curvature is convex or concave, the sufficient condition can be a useful guide. However, if F_{XX} is near but on the wrong side of zero, then one must calculate the value to establish stability (although in any case the adiabatic sound speed would exceed the speed of light in these ambiguous cases). One can apply Eqs. (27)-(28) to the $F(X)$ plots in the examples of the following section to verify their usefulness as a quick indicator of the stability of possible kinetic k-essence Lagrangians.

To clarify further possible degeneracies between k-essence and quintessence physics, we examine in the next section specific examples that illustrate the points made above.

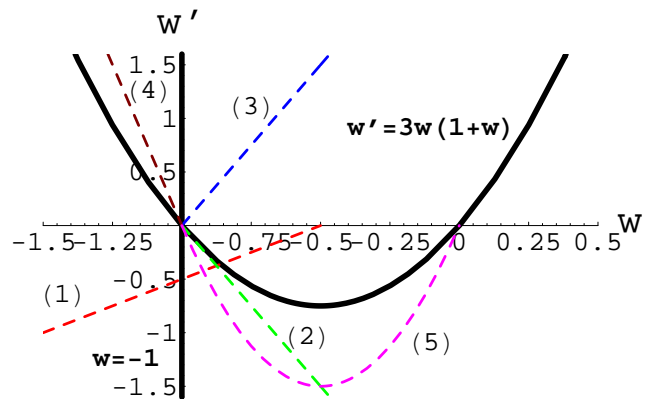


FIG. 2: We investigate for a number of equations of state whether or not they can be described by a kinetic k-essence model. The plot shows the trajectories in the w - w' plane for some examples. It also shows the boundaries between stable and unstable k-essence regions (see Fig. 1): the lines $w = -1$ and $w' = 3w(1+w)$. The equations of state are: (1) $w = -0.9 + 0.4(1-a)$, see §VIB; (2) $1+w = (1-0.9)a^{-3}$, (3) $1+w = (1-0.9)a^3$, (4) $1+w = (1-1.1)a^{-6}$, all §VIC; (5) the Chaplygin gas, see §VID. Note that $w = \text{const}$ equations of state (see §VIA) correspond to points on the horizontal axis.

VI. EXAMPLES

We here consider a few particular ansatzes for the DE equation of state $w(a)$ and analyze their degree of degeneracy with a (stable) k-essence solution. To begin, we check whether the equation of state lies today ($z = 0$) in one of the allowed regions given by Eq. (18). If so, a function $F(X)$ can be constructed to reproduce this equation of state in a certain redshift range around $z = 0$. If $w(a)$ stays in the allowed region for all a , then we can find an $F(X)$ degenerate with this dynamics for all redshifts. (But it is important to note that it will not be equivalent to a quintessence field because the sound speed will not be unity; the two different physical models will be distinguishable to the extent that spatial inhomogeneities in the dark energy component are relevant.)

If $w(a)$ leaves the allowed regions after some time, then we can only match stable k-essence with quintessence over the redshift range where $w(a)$ does lie in one of the allowed regions. Since in practice we can only measure $w(a)$ in a limited redshift range anyway, the relevant question is whether we can reproduce a given equation of state in the redshift range constrained well by data, not necessarily for all redshifts. For each sample equation of state considered below, we establish the acceptable redshift range and evaluate the equivalent Lagrangian function $F(X)$ and sound speed c_s^2 . The examples are designed to lie in different regions of the phase space with different stability properties, for illustration; see Fig. 2.

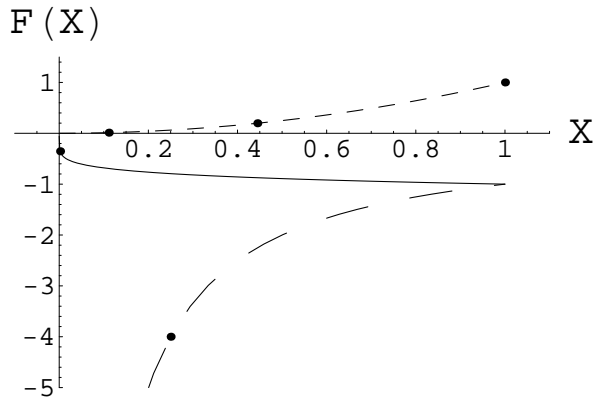


FIG. 3: $F(X)$ for constant w (ansatz A) with $w = -1.5$ (solid), $w = -1/3$ (long dashed) and $w = +1/3$ (short dashed). For $w < 0$, we choose $X(z=0) = 1$ to be at the end of the plotted domain; the field evolves with time from left to right and the marked points correspond to $z = 1$. For $w > 0$, the field moves toward $X = 0$ and the markers on the $w = +1/3$ curve indicate from left to right $z = 0, 1, 2$. The Lagrangians for negative constant w ($\neq -1$) correspond to unstable solutions. An equation of state $w \equiv -1$ can be obtained from any $F(X)$ with an extremum by letting X sit at that extremum (or from a cosmological constant $F = \text{const}$).

A. Constant w

A constant equation of state is simple and approximates some well known components. For example, radiation has $w = +1/3$, matter $w = 0$ and a cosmological constant $w = -1$. Moreover, a (canonical) free field theory, with zero self-interaction potential $V = 0$, where the Lagrangian is simply $F(\phi, X) = X$, has $w = +1$.

It is useful to discuss the cases $w = -1$ and $w \neq -1$ separately. From Eq. (6), there are two ways to obtain $w = -1$. The first is by having $X = \text{const}$. Since constant X implies constant ρ , such solutions can *only* give $w = -1$. From Eq. (9), $X = \text{const}$ is a solution if $F_X = 0$. These solutions are special because since $dX/da = 0$ we cannot use Eq. (18) for the sound speed; instead we have to go back to Eq. (14). We see that (under the assumption that $F_{XX} \neq 0$, otherwise we have a canonical scalar field) this case has $c_s^2 = 0$ and is thus marginally stable. If X is not constant, the only way to get $w \equiv -1$ is by having $F = \text{const}$. This is just a cosmological constant and the adiabatic sound speed is not defined because there are no perturbations.

Next we look at $w \neq -1$. One can obtain constant $w \neq -1$ solutions if X is not constant and the functional form of $F(X)$ has the right form. Since constant equations of state are points on the horizontal axis in figure 1, we need $w \geq 0$ for the equation of state to be in the stable region. More quantitatively, from Eq. (18), we have

$$c_s^2 = w. \quad (29)$$

In other words, any solution leading to a constant neg-

ative equation of state ($\neq -1$) is unstable viewed as k-essence.

It is straightforward to explicitly construct k-essence Lagrangian densities corresponding to constant w without applying the machinery developed in the previous section (see also [33]). Instead, we can just use Eq. (6) to get

$$XF_X = \frac{1+w}{2w}F, \quad (30)$$

which integrates to

$$F(X) \propto X^{\frac{1+w}{2w}}. \quad (31)$$

Since the energy density should be positive, F must be positive for the $w > 0$ solutions and negative for the $w < 0$ ones. As discussed in §III, we are free to choose the magnitude of the proportionality factor.

Let us now look at some specific cases. First of all, it is useful to note that we cannot reproduce matter-like behavior, as for $w = 0$ the k-essence Lagrangian (31) is not well defined. This result makes sense: $w = 0$ corresponds to a pressure-less material, whereas in the case of k-essence, the pressure is the same as the Lagrangian density. In other words, $w = 0$ would mean that the Lagrangian density is zero everywhere, which of course just means there is no DE model at all.

Equation (31) also confirms that $w = 1$ corresponds to a canonical free field (no potential) Lagrangian $F(X) = X$ (which is simultaneously a k-essence and a quintessence model). Note that skating models (see [3] and references therein), moving along a constant potential with $w' = -3(1-w^2)$, stretch between true free fields $V = 0$, $w = 1$ and pure constant potentials $V = V_0$, $w = -1$, following $X \sim a^{-6}$. This leads by Eq. (9) to $F_X = \text{const}$, or $F \sim X + \text{const}$. In this sense one can think of kinetic k-essence models as skaters that “push off”, altering their kinetic energy. Radiation-like behavior, $w = 1/3$, is generated by a function of the form $F(X) = X^2$. (Note this should not be interpreted as a perturbation around a minimum of a potential since the field motion does not correspond to rolling in $F(X)$.) Figure 3 illustrates several examples.

B. Time variation: $w(a) = w_0 + w_a(1-a)$

A commonly used ansatz allowing for time variation is $w(a) = w_0 + w_a(1-a)$, which provides a good fit to many canonical scalar field and other model behaviors [34], at least over the past expansion history. Since $w' = -w_a a = w - (w_0 + w_a)$, in the w - w' plane the equation of state starts in the past somewhere on the w -axis (at $w(a \ll 1) = w_0 + w_a$). Thus from the previous subsection we know it cannot be wholly degenerate with a stable k-essence model. Its dynamics corresponds to a straight line with slope one that may cross through the stable region but again in the far future lie in an unstable region.

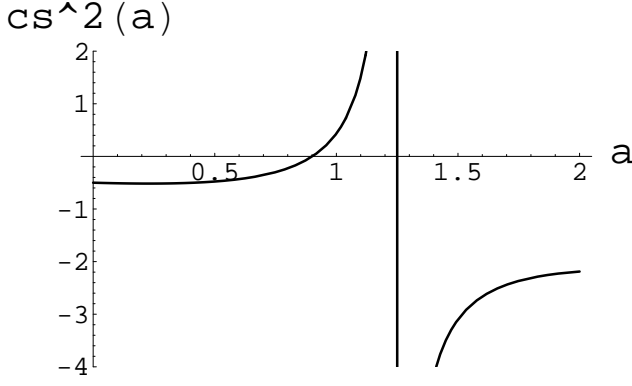


FIG. 4: Ansatz B with $(w_0, w_a) = (-0.9, 0.4)$. The adiabatic sound speed squared c_s^2 is plotted vs. the scale factor a , clearly showing that $c_s^2 > 0$ today but becomes negative both at $a \approx 0.9$ and $a \approx 1.25$. Hence, this equation of state can only be described by a stable kinetic k-essence solution in the (rather limited) range between those times (also see Fig. 5).

Thus there is only a finite range of redshift when it may look like a stable kinetic k-essence model. The values of w and w' today are given by $w(a=1) = w_0$ and $w'(a=1) = -w_a$. For a model within the stable region today we consider $(w_0, w_a) = (-0.9, 0.4)$, in Fig. 4. This crosses the $w' = 3w(1+w)$ line at $a \approx 0.9$. This means that when calculating $F(X)$, starting at $a = 1$, it will become double valued at $a \approx 0.9$. In the future, the equation of state crosses the other boundary $w = -1$ at $a \approx 1.25$. Hence, this equation of state is only degenerate with a stable kinetic k-essence model in the very limited range $a \approx 0.9 - 1.25$.

To calculate the k-essence Lagrangian explicitly, we need to solve Eq. (24). This can be done analytically in the case at hand, giving

$$X(a) = C a^{-6(w_0+w_a)} [1 + w_0 + w_a(1-a)]^2 e^{6w_a a}, \quad (32)$$

where C is a positive constant. In the region between its extrema (the points where w crosses one of the boundaries in Fig. 1), $X(a)$ can be uniquely inverted and $F(X)$ can be computed after using Eq. (25) to find $F(a)$. The solution is shown in Fig. 5 for $(w_0, w_a) = (-0.9, 0.4)$. Note how $F(X)$ becomes double valued when going beyond the extrema.

C. Thawing/Freezing Regions

The ansatz of $w' = x(1+w)$ leads to an equation of state

$$1 + w(a) = (1 + w_0) a^x. \quad (33)$$

This describes a thawing model for $x > 0$, where the equation of state starts frozen at high redshift so $w = -1$, and then begins rolling away from it. This form describes

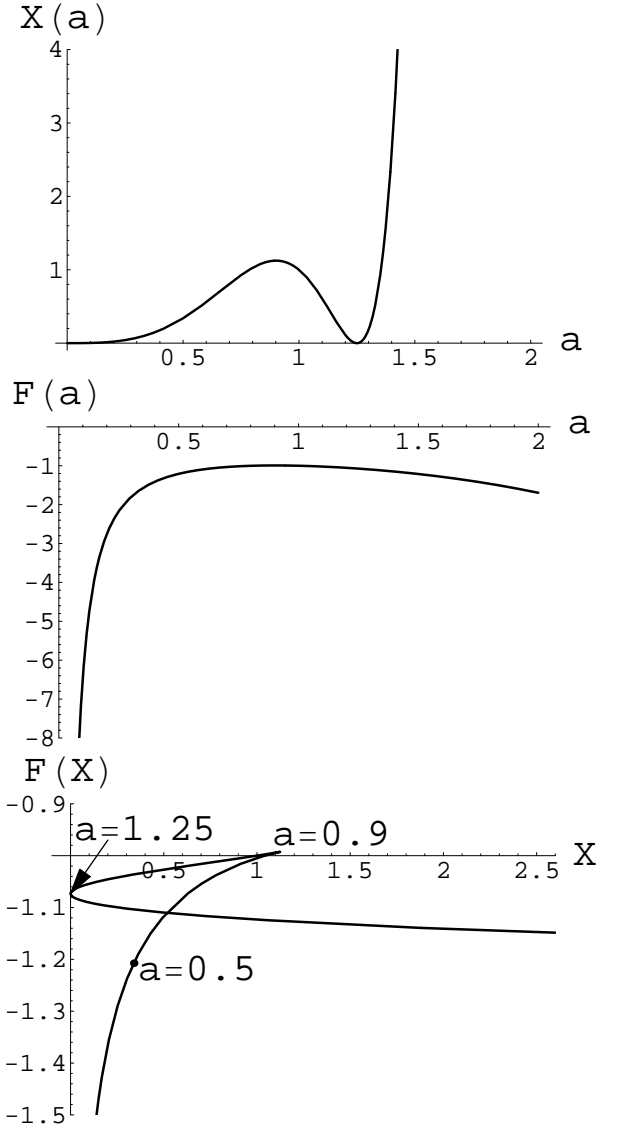


FIG. 5: As Fig. 4, but plotting $X(a)$ and $F(a)$ (see text and Fig. 4 for details), normalized such that today $X = 1$ and $F = -1$. The third panel shows the function $F(X)$ obtained from $X(a)$ and $F(a)$ over the range $a = 0.1 - 1.5$. $F(X)$ turns around at the points where $dX/da = 0$ (i.e. where c_s^2 becomes negative, at $a \approx 0.9$ and $a \approx 1.25$) and is therefore not single valued. Hence, this $(w_0, w_a) = (-0.9, 0.4)$ example model only corresponds to a well defined $F(X)$ in the region $a \approx 0.9 - 1.25$.

well a number of renormalizable power law potentials and pseudo-Nambu Goldstone boson (PNGB) models. For $x < 0$ it is simply a toy model of an equation of state approaching (freezing into) a cosmological constant state. See [1] for more on thawing and freezing models.

From Eq. (18), the adiabatic sound speed for these models is given by

$$c_s^2(a) = w(a) - \frac{x}{3}. \quad (34)$$

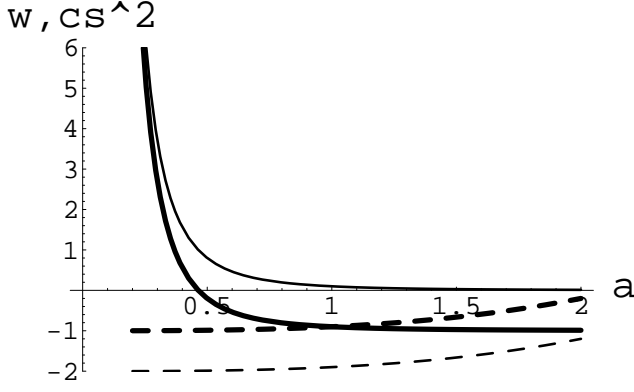


FIG. 6: Ansatz C with $w_0 = -0.9$ and $x = -3$ (solid) or $x = 3$ (dashed). The equation of state (thick lines) and adiabatic sound speed squared (thin lines) are plotted vs. a . Note $x = 3$ lies in a forbidden region ($c_s^2 < 0$) for all a and therefore cannot correspond to a stable kinetic k-essence solution. For $x = -3$, the corresponding k-essence Lagrangian is exhibited in Fig. 7.

This shows that c_s^2 is positive today only if $x < 3w_0$. It is positive at all times if $w_0 > -1$ and $x < -3$ (for $w_0 < -1$, it can only lie in the stable region for a limited redshift range regardless of the value of x). Still, we see that the criterion that c_s^2 be positive already rules out all the thawing equations of state and part of the freezing ones.

To construct $F(X)$, we again solve Eq. (24), giving

$$X(a) = C a^{2x+6} e^{-\frac{6(1+w_0)}{x} a^x}, \quad (35)$$

where C is a positive constant. Note that if $(x+3)/(1+w_0) > 0$ this function has an extremum at $a = \{(x+3)/[3(1+w_0)]\}^{1/x}$. After finding $F(a)$ from Eq. (25), we can again calculate $F(X)$.

We explore three different cases, corresponding to diverse physical situations. The example with $x = 3$ describes well a thawing equation of state, evolving away from cosmological constant behavior, but would be an unstable k-essence model. For $x = -3$, the dynamics is freezing, approaching a cosmological constant, and the corresponding k-essence model has positive sound speed squared for all redshifts. The model with $x = -6$ and $w_0 = -1.1$ gives a toy phantom model which approaches a cosmological constant, and lies in the stable region only at recent redshifts. These cases are illustrated in Figs. 6-9.

D. Generalized Chaplygin Gas

Dark energy that behaves like matter ($w = 0$) at early times and like a cosmological constant ($w = -1$) at late

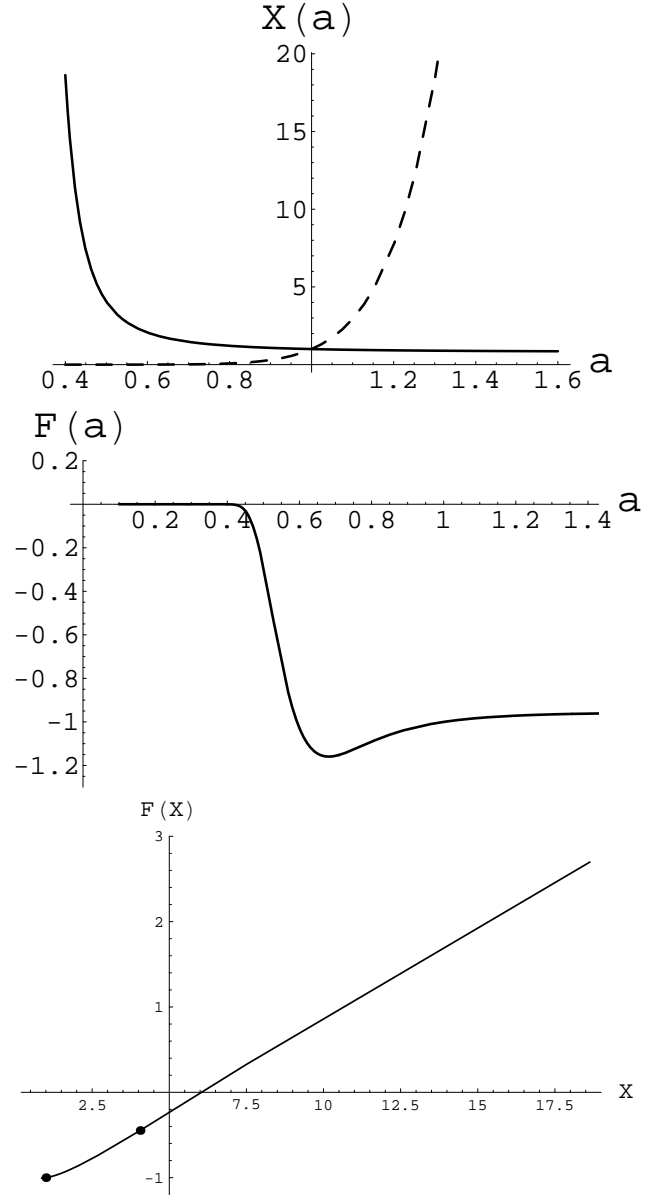


FIG. 7: As Fig. 6 but showing the k-essence functions $X(a)$, $F(a)$, and $F(X)$. In the last panel $F(X)$ is plotted for the stable $x = -3$ case over $a = 0.4 - 1.6$ (from top right to bottom left), with markers at $z = 1$ (right) and $z = 0$ (left).

times can be described by an equation of state

$$1 + w(a) = \left[1 - \frac{w_0}{1 + w_0} a^{3(n+1)} \right]^{-1}, \quad (36)$$

where we take $w_0 \in (-1, 0)$ and $n > 0$. A model that produces exactly this equation of state is the generalized Chaplygin gas (GCG, [13]), which can also be described by a perfect fluid/gas with equation of state $p = -A/\rho^n$, where A is a constant. The original model, the Chaplygin gas ([35, 36]), corresponds to the case $n = 1$. The GCG has been extensively studied in hope of providing a

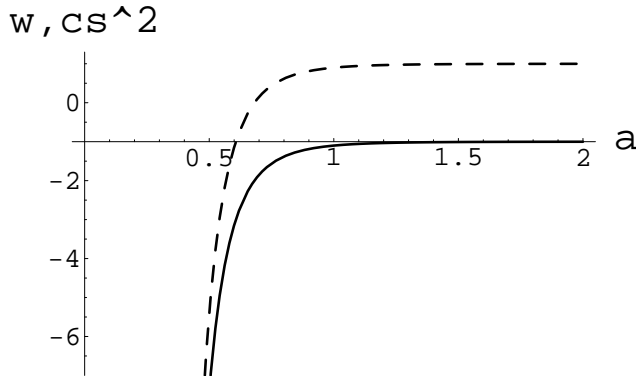


FIG. 8: As Fig. 6, but for the phantom case $w_0 = -1.1$ and $x = -6$. Here the solid curve shows $w(a)$ and the dashed curve c_s^2 .

unified description of dark matter and dark energy, but in such an approach it has problematic issues with structure formation [14, 15, 16]. However, it is still interesting to study as a dark energy model. This equation of state can be reproduced by a k-essence model (see, e.g., [13]) and, of the dynamical equations of state we discuss in this paper, this is the only one where an *explicit* expression can be found for the k-essence Lagrangian density $F(X)$. The Lagrangian for $n = 1$ can be linked to the tachyon in string theory and to the dynamics of branes (as referred to in §II).

From the expression for $w(a)$ one finds that its dynamical trajectory is a parabola given by $w' = 3(n+1)w(1+w)$, with $w(a)$ evolving from 0 to -1 (also see the mock model in [37]). For $n > 0$ this equation of state always lies completely in the allowed region B. It is therefore possible to reproduce it with a stable kinetic k-essence solution. Note that in the limit $n \rightarrow 0$ we approach the boundary of the allowed regions, the constant pressure line. (For $n > 1$ there will be epochs where the trajectory crosses the null line given by Eq. (19); see §IV.) We consider two examples ($n = 0.5, 1$ with $w_0 = -0.9$) in Fig. 10.

To construct $F(X)$, we first solve equation (24) to find

$$X(a) = \left(1 - \frac{w_0}{1+w_0} a^{3(n+1)}\right)^{-\frac{2n}{n+1}}, \quad (37)$$

where we have chosen the normalization such that X goes from 1 at $a = 0$ to 0 at $a = \infty$. The expression can easily be inverted to give $a(X)$. Subsequently, Eq. (26) leads to

$$F(X) = -A^{\frac{1}{n+1}} (1 - X^{\frac{n+1}{2n}})^{\frac{n}{n+1}} \quad (38)$$

(cf. [13]), where A is the constant appearing in the GCG equation $p = -A/\rho^n$. We plot two examples in Fig. 11.

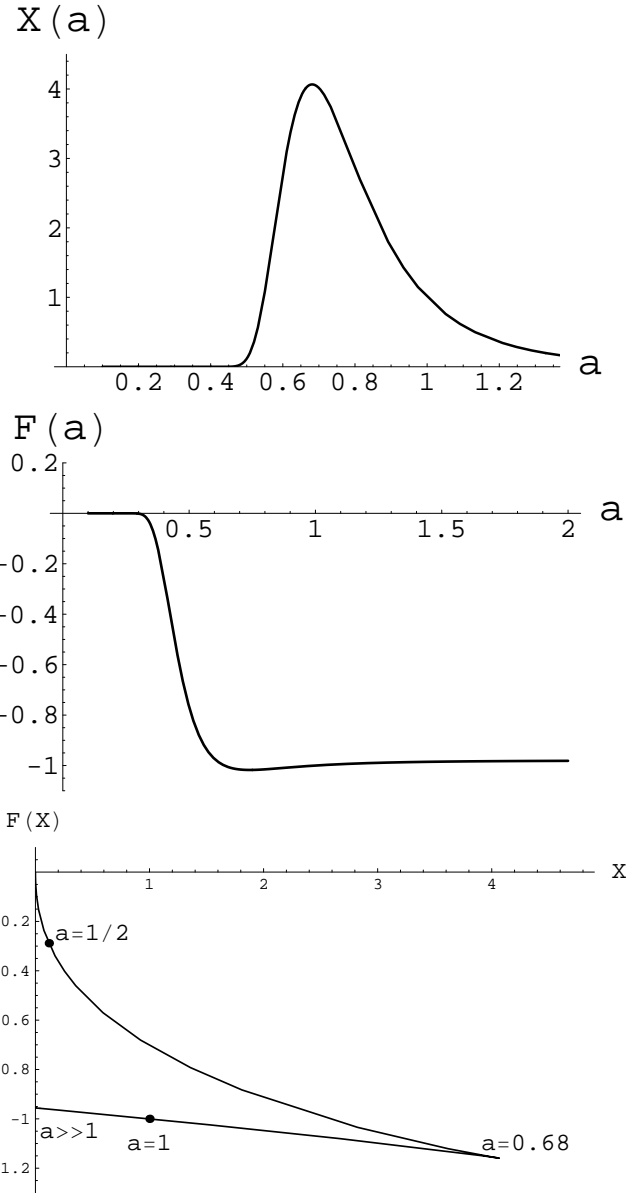


FIG. 9: As Fig. 8, but showing the functions $X(a)$, $F(a)$, and $F(X)$.

VII. CONCLUSIONS

Kinetic k-essence is in some sense an equally probable solution to the dark energy conundrum as quintessence, trading a single potential function $V(\phi)$ for a single kinetic function $F(X)$. Similarly, one can find equivalent motivations for it from quantum field and extra dimensional theories.

We have established stability regions for such models within the equation of state phase space, based on the necessary condition of a non-imaginary sound speed. Conversely, we find closed form solutions, given some equation of state $w(a)$, for a dynamically correspond-

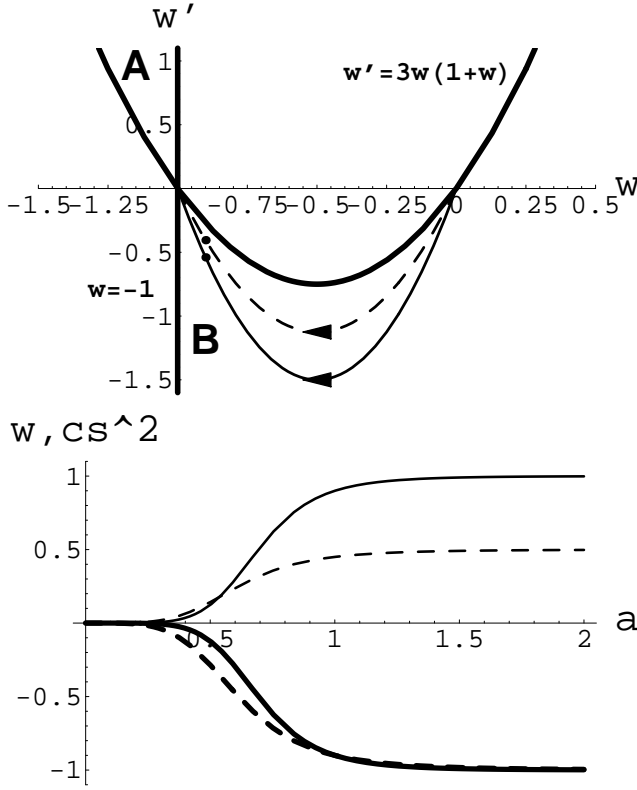


FIG. 10: Ansatz D (the generalized Chaplygin gas) with $w_0 = -0.9$ and $n = 1$ (solid) or $n = 0.5$ (dashed). The first panel shows the trajectories in the w - w' plane. The direction of increasing scale factor is indicated by arrows and dots mark the values of w and w' today. Since the equations of state lie completely in region B, they can be obtained from purely kinetic k-essence Lagrangians (see Fig. 11). The second panel shows the equation of state w (thick lines) and the adiabatic sound speed squared c_s^2 (thin lines) as a function of scale factor a .

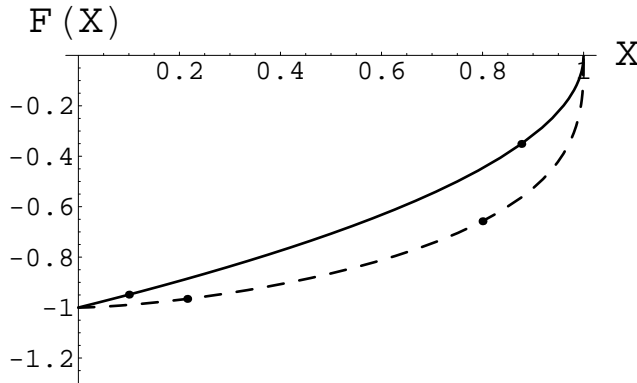


FIG. 11: As Fig. 10, but plotting the corresponding k-essence Lagrangian density $F(X)$ (with F in units of $A^{1/(n+1)}$). X starts at one and then moves to zero as the scale factor increases. Markers indicate the points where $z = 1$ (right) and $z = 0$ (left).

ing kinetic k-essence Lagrangian $F(X)$. Eqs. (11-13) and (27-28) constitute a simple prescription – “by eye” – for characterizing kinetic k-essence theories.

Using these results, we investigated the limits to dynamical degeneracy between kinetic k-essence and quintessence. Analyzing four types of equations of state representing diverse dynamics, we found the limits in redshift defining the degeneracy region. For several equation of state models, both $w > -1$ and phantom, this implies that one could rule out all kinetic k-essence models with a sufficient redshift range of measurements. On the other hand, an equation of state similar in form to the generalized Chaplygin gas could be equally described by k-essence for all redshifts (as was already known). This clear definition of degeneracy regions offers increased hope that with future observational data on the dark energy dynamical and microphysical effects we can discern which approach describes the new physics behind our accelerating universe.

Acknowledgments

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Appendix

As an example, consider the simple case of a 3-brane embedded in a $4+1$ dimensional space-time with a fixed Minkowski metric η_{ij} , $i, j = 0, \dots, 4$. The brane can be parametrized by four world-sheet coordinates x^μ ($\mu = 0, \dots, 3$) so that the location of the brane in space-time (or target space) is given by $X^i = X^i(x^\mu)$ ($i = 0, \dots, 4$). In general, the Nambu-Goto action of a p -brane is given by the volume of its world-sheet

$$S_{\text{NG}} = \int d^4x \sqrt{-\det(g_{\mu\nu})}, \quad (39)$$

where G_{ij} is the target space metric and $g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j G_{ij}$ is the induced metric on the brane. In our example, $G_{ij} = \eta_{ij}$.

In the static gauge, the world-sheet coordinates are chosen to coincide with the first four target space coordinates: $x^\mu = X^\mu$ for $\mu = 0, 1, 2, 3$. We now call the fifth target space coordinate $X^4 = \phi$ to emphasize that it is a scalar from the world-sheet point of view. In this parametrization/gauge, $g_{\mu\nu} = \eta_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi$. If we now assume that ϕ only depends on the time coordinate $t = x^0$, we get the Lagrangian

$$S = \int d^4x \sqrt{1 - 2\dot{\phi}^2}. \quad (40)$$

This is equivalent to the Chaplygin gas Lagrangian (see also §VID) if there we assume a fixed Minkowski background. We can reproduce Lagrangians of the more general form $\mathcal{L} = -V(\phi)\sqrt{1 - 2\dot{\phi}^2}$, with a variety of $3+1$ dimensional backgrounds, by having different $4+1$ dimensional backgrounds G_{ij} . The “potential” $V(\phi)$ then corresponds to a warp factor in the metric.

We wish to stress that this discussion is merely meant to show that there are physical ways to get k-essence Lagrangians. We do not claim that the above constitutes a realistic cosmological model.